

STAT 345 - Handout 5
Estimating the CDF of a $N(0, 1)$
BASED ON SECTION: 4.6

Consider finding the cumulative probability of a $N(0, 1)$ at a point $c > 0$, i.e.

$$P(Z \leq c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^c e^{-\frac{z^2}{2}} dz \text{ where } c > 0$$

One way to numerically estimate this is to use the Taylor series expansion of $e^{-\frac{z^2}{2}}$ as follows:

$$\begin{aligned} P(Z \leq c) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^c e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-\frac{z^2}{2}} dz + \frac{1}{\sqrt{2\pi}} \int_0^c e^{-\frac{z^2}{2}} dz \\ &= 0.5 + \frac{1}{\sqrt{2\pi}} \int_0^c e^{-\frac{z^2}{2}} dz \end{aligned}$$

Now

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ letting } x = \frac{-z^2}{2} \\ \Rightarrow e^{-\frac{z^2}{2}} &= \sum_{n=0}^{\infty} \frac{\left(\frac{-z^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^n n!} \\ \Rightarrow \int_0^c e^{-\frac{z^2}{2}} dz &= \int_0^c \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2^n n!} dz \end{aligned}$$

The integral on the right can be approximated by taking a finite sum, say up to N where N is large:

$$\begin{aligned} \int_0^c e^{-\frac{z^2}{2}} dz &\approx \int_0^c \sum_{n=0}^N \frac{(-1)^n z^{2n}}{2^n n!} dz \\ &= \sum_{n=0}^N \frac{(-1)^n}{2^n n!} \int_0^c z^{2n} dz \\ &= \sum_{n=0}^N \frac{(-1)^n c^{2n+1}}{2^n n! (2n+1)} \end{aligned}$$

Consider $c = 0.50$. The table below gives the steps in estimating $P(0 < Z \leq 0.50)$ for various values of N .

n/N	$\frac{(-1)^n(0.5)^{2n+1}}{2^n n!(2n+1)}$	$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^N \frac{(-1)^n(0.5)^{2n+1}}{2^n n!(2n+1)}$
0	0.5	0.199471033962926
1	-0.0208333333	0.191159740881137
2	0.00078125	0.191471414371704
3	-2.32515E-05	0.191462138374961
4	5.65140E-07	0.191462363833215
5	-1.15597E-08	0.191462359221569
6	2.03777E-10	0.191462359302864
7	-3.15368E-12	0.191462359301606
8	4.34791E-14	0.191462359301623
9	-5.40311E-16	0.191462359301623
10	6.11066E-18	0.191462359301623
11	-6.34011E-20	0.191462359301623
12	6.07594E-22	0.191462359301623
13	-5.40949E-24	0.191462359301623
14	4.49680E-26	0.191462359301623
15	-3.50557E-28	0.191462359301623
16	2.57274E-30	0.191462359301623
17	-1.78363E-32	0.191462359301623
18	1.17168E-34	0.191462359301623
19	-7.31309E-37	0.191462359301623
20	4.34772E-39	0.191462359301623
21	-2.46756E-41	0.191462359301623
22	1.33971E-43	0.191462359301623
23	-6.97121E-46	0.191462359301623
24	3.48264E-48	0.191462359301623
25	-1.67303E-50	0.191462359301623