.1.2 1/5

CH 8 CONFIDENCE INTERVALS FOR A SINGLE SAMPLE.

8.2 CI ON MEAN OF NORMAL DISTRIBUTION, KNOWN VARIANCE.

EX SOFT DRINK CANS ARE FILLED BY AN AUTOMATED FILLING MACHINE. ASSUME THAT THE FILL VOLUMES OF CANS ARE INDEPENDENT, NORMAL RVS WITH MEAN $\mu = 12.1$ fl. oz. AND STD DEV $\sigma = .5$ fl. oz.

a) WHAT IS THE DISTRIBUTION OF THE AVERAGE FILL VOLUME OF $n=100$ CANS?

b) WHAT IS THE PROBABILITY THE AVERAGE FILL VOLUME IS LESS THAN 12 fl. oz?

c) " " " " " " " BETWEEN 11.9 AND 12.3 oz?

X_i = FILL VOLUME OF CAN i.

a) $X_1, \dots, X_{100} \overset{\text{iid}}{\sim} N(12.1, .5^2) = N(\overset{\mu}{12.1}, \overset{\sigma^2}{.25})$

$\bar{X} = \frac{1}{100}(X_1 + \dots + X_{100}) \sim N(12.1, .25/100) = N(\overset{\mu_{\bar{X}}}{12.1}, \overset{\sigma^2_{\bar{X}}}{.0025})$

~~b) $P_r(\bar{X} < 12) = P_r($~~

b) $P_r(\bar{X} < 12) = P_r\left(\frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} < \frac{12 - 12.1}{\sqrt{.0025}}\right) = P_r(Z < -2) = .02275$

$\approx 2.3\%$.

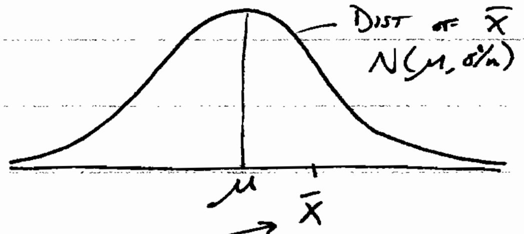
c) $P_r(11.9 < \bar{X} < 12.3) = P_r\left(\frac{11.9 - 12.1}{\sqrt{.0025}} < \frac{\bar{X} - \mu_{\bar{X}}}{\sigma_{\bar{X}}} < \frac{12.3 - 12.1}{\sqrt{.0025}}\right)$

$= P_r(-4 < Z < 4) = .99967 - .000033 = .999934 \approx 99.99\%$.

2/5 Suppose $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with μ unknown and σ^2 known.

We would like to estimate μ with a certain degree of certainty.

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

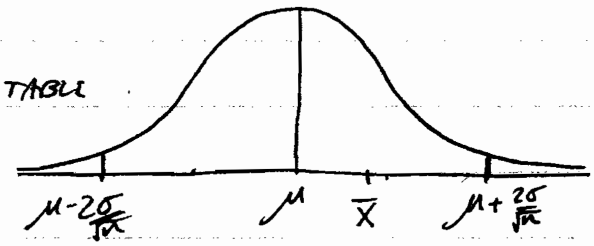


If this $\bar{X}$ is from our 1st sample, how good is it?

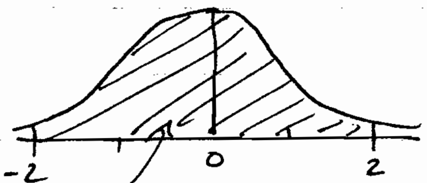
Suppose if $\bar{X}$ falls within 2 STD^s of the mean μ, we are satisfied.

Using 68-95-99.7 rule or table

About 95% of all samples will have $\bar{X}^s$ in this region



Recall $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$



95% of standardized means are here

$$Pr(-2 < Z < 2) = .9545$$

$$Pr\left(-2 < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < 2\right) = .9545$$

$$Pr\left(-\frac{2\sigma}{\sqrt{n}} < \bar{X} - \mu < \frac{2\sigma}{\sqrt{n}}\right) = .9545$$

$$Pr\left(\mu - \frac{2\sigma}{\sqrt{n}} < \bar{X} < \mu + \frac{2\sigma}{\sqrt{n}}\right) = .9545$$ – what we said above, $\bar{X}$ within 2 SSD^s of μ.

Rewrite as

$$Pr\left(\bar{X} - \frac{2\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{2\sigma}{\sqrt{n}}\right) = .9545$$

unknown μ

The confidence interval (CI) at 95% for the unknown mean μ is

Roughly $\bar{X} - \frac{2\sigma}{\sqrt{n}} \le \mu \le \bar{X} + \frac{2\sigma}{\sqrt{n}}$.

More precisely $\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$ or $\bar{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$.

3/5

EX SUPPOSE THAT THE MEAN OF THE 100 CANS WAS ACTUALLY $\bar{X} = 12.04$ FL.OZ. FIND A 95% CONFIDENCE INTERVAL (CI) FOR THE MEAN μ.

$$VAR(\bar{X}) = \frac{\sigma^2}{n} = .0025$$

95% CI FOR μ: $\bar{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + 1.96\frac{\sigma}{\sqrt{n}}$

$$12.04 - 1.96\sqrt{.0025} \le \mu \le 12.04 + 1.96\sqrt{.0025}$$

$$11.942 \le \mu \le 12.138$$

INTERPRETATION: THE RANDOM INTERVAL (11.942, 12.138) CONTAINS μ WITH 95% CONFIDENCE.

OR, IN 95% OF RANDOM SAMPLES, THE INTERVAL OBTAINED WILL CONTAIN μ.

HANDOUT 7]

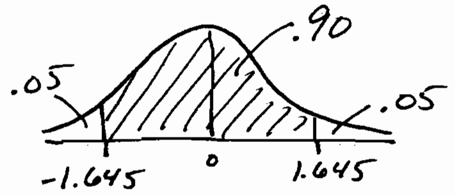
NOTE: IT IS TEMPTING TO CONCLUDE THAT μ IS IN THE INTERVAL WITH PROBABILITY .95. HOWEVER, μ IS FIXED AND UNKNOWN, AND THE INTERVAL IS RANDOM AND KNOWN. ONCE THE INTERVAL IS CONSTRUCTED, THE INTERVAL EITHER CONTAINS OR DOESN'T CONTAIN μ. YOU ARE EITHER RIGHT OR WRONG. THIS IS WHY THE ABOVE INTERPRETATION IS CORRECT.

NOTES

1. CAN COMPUTE ANY SIZED CONFIDENCE INTERVAL.

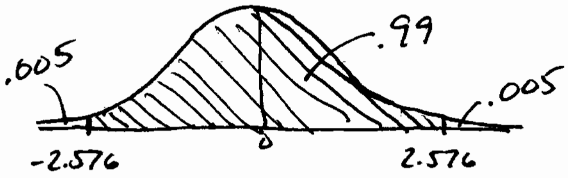
EX. 90% CI FOR μ:

$$\bar{X} \pm 1.645\frac{\sigma}{\sqrt{n}}$$



EX. 99% CI FOR μ:

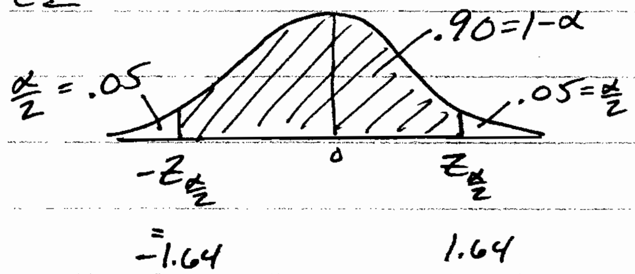
$$\bar{X} \pm 2.576\frac{\sigma}{\sqrt{n}}$$



4/9

IN GENERAL, $100(1-\alpha)\%$ CI

TABLE II | Φ | OR IV



EX $90\% \Rightarrow 1-\alpha = .9$

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

EX $80\% \Rightarrow 1-\alpha = .8$

$\alpha = .2$, $\frac{\alpha}{2} = .1$

$Z_{\frac{\alpha}{2}} = 1.28$

2. CI CAN BE WRITTEN AS $\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$$\text{ESTIMATE} \pm \text{MARGIN-OF-ERROR (ME)}$$

CAN FIX MARGIN-OF-ERROR TO SELECT SAMPLE SIZE.

σ EITHER KNOWN OR CAN GUESS (OR ESTIMATE).

HAVE CONTROL OF α AND n.

$$n = \left(\frac{Z_{\frac{\alpha}{2}}\,\sigma}{(ME)}\right)^2 \text{ ROUNDED UP} \nearrow.$$

3. ~~ERROR~~ WITH EVERYTHING ELSE HELD CONSTANT, THE WIDTH OF THE CI WILL:

INCREASE AS α INCREASES → INCREASED CONFIDENCE

INCREASE AS σ INCREASES → INCREASED VARIABILITY

DECREASE AS n INCREASES. → INCREASED INFORMATION.

4. TYPICAL VALUES OF α ARE

	α	C	$Z_{\frac{\alpha}{2}}$
	.1	90%	1.645
★	.05	95%	1.960
	.01	99%	2.576

EX. BIOLOGISTS STUDYING THE HEALING OF SKIN WOUNDS MEASURED THE RATE AT WHICH NEW CELLS CLOSED ON A RAZOR CUT MADE IN THE SKIN OF AN ANESTHETIZED ~~N~~ NEWT. THE DATA FOR $n=18$ NEWTS IN MICROMETERS / HOURS:

29 27 34 40 22 28 14 35 26

35 12 30 23 18 11 22 23 33

ASSUMING THE NEWTS ARE A RANDOM SAMPLE FROM THEIR SPECIES, AND THAT THE POPULATION STANDARD DEVIATION IS 8 mm/HR, ~~FIND~~

FIND A 90% CI FOR THE MEAN HEALING RATE OF NEWTS.

1. STEM PLOT

1	1 2 4 8
2	2 2 3 3 6 7 8 9
3	0 3 4 5 5
4	0

ROUGHLY SYMMETRIC, UNIMODAL.
LOOKS FAIRLY NORMALLY DISTRIBUTED.

μ = MEAN HEALING RATE OF POPULATION OF NEWTS

$\sigma = 8$ mm/HR , $n = 18$.

90% CI $\Rightarrow Z_{\frac{\alpha}{2}} = 1.645$

$\bar{X} = \frac{1}{18}(29+27+34+\cdots+33) = 25.67$ mm/HR.

90% CI FOR μ: $\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

$25.67 \pm 1.645\left(\frac{8}{\sqrt{18}}\right)$

25.67 ± 3.10

$(22.57, 28.77)$

THE INTERVAL (22.57, 28.77) CONTAINS μ WITH 90% CONFIDENCE.

8.3 1/3

8.3 CI ON MEAN OF NORMAL DISTRIBUTION, VARIANCE UNKNOWN.

PREVIOUSLY, WE ASSUMED σ WAS KNOWN.

IF WE KNOW σ, WE KNOW μ!

TYPICALLY σ IS <u>UNKNOWN</u>.

IF THE POPULATION STANDARD DEVIATION IS UNKNOWN, AND THE SAMPLE SIZE IS LARGE ($n > 40$), THEN BY THE CENTRAL LIMIT THEOREM (CLT)

$\bar{X} \overset{a}{\sim} N(\mu, \sigma^2/n)$ APPROXIMATELY.

ALSO $S \to \sigma$ AS $n \to \infty$.

SINCE σ UNKNOWN, USE S INSTEAD.

SO $\dfrac{\bar{X} - \mu}{S/\sqrt{n}} \overset{a}{\sim} N(0,1)$.

BUT, IF THE SAMPLE SIZE IS SMALL ($n \le 40$), THE SAMPLING DISTRIBUTION OF $\bar{X}$ LOOKS NORMAL, BUT HAS INCREASED VARIABILITY: THE t DISTRIBUTION.

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$$

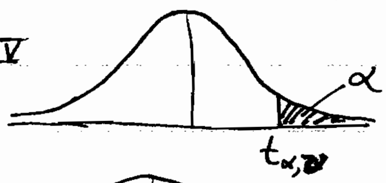
$n-1$ = degrees-of-freedom = df.

THERE ARE AN INFINITE NUMBER OF t DISTRIBUTIONS INDEXED BY df.

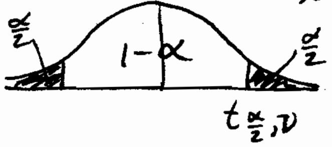
A $100(1-\alpha)\%$ CI OF μ IS

$$\bar{x} \pm \underbrace{t_{\frac{\alpha}{2}, n-1}}_{\text{NOW } t \text{ INSTEAD OF } z_{\frac{\alpha}{2}}} \frac{s}{\sqrt{n}}$$

TABLE IV



IN OUR CONTEXT OF A CI, WE WANT



SO BE CAREFUL!

EX.

n	C	$t_{\frac{\alpha}{2}, n-1}$
15	90%	$t_{.05,14} = 1.761$
23	95%	$t_{.025,22} = 2.074$
5	99%	[illegible]

IF df NOT IN TABLE, ROUND DOWN TO BE CONSERVATIVE WITH A WIDER INTERVAL [illegible]

2/3

General Forms for CI's:

	Normal, Known σ	Large Sample Normal Unknown σ $n \geq 40$	Small Sample Normal Unknown $n < 40$	Large Sample Population Proportion $np > 5$ and $n(1-p) > 5$
Parameter of Interest : PAR	μ	μ	μ	p
Estimate of Parameter : EST	$\bar{X}$	$\bar{X}$	$\bar{X}$	$\hat{p}$
Standard Error of Est : SE(EST)	$\frac{\sigma}{\sqrt{n}}$	$\frac{s}{\sqrt{n}}$	$s/\sqrt{n}$	$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Distribution : $\frac{EST - PAR}{SE(EST)} \sim DIST$	$N(0,1)$	$N(0,1)$	t_{n-1}	$N(0,1)$

$$EST \pm C^{*} \cdot SE(EST)$$

Cut off from distribution; either $z_{\frac{\alpha}{2}}$ or $t_{\frac{\alpha}{2}, n-1}$

[Handout 8]

Assumptions to use small sample ($n \leq 40$) t procedures:

$X_1, \ldots, X_n$ ~~is~~ are iid and roughly normal : symmetric, unimodal, no outliers (small deviations from normality are expected in small samples).

Check normality with stem-and-leaf plot.

~~or~~

Why the t-distribution? (Not on exam)

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \text{ if } \sigma \text{ is known}$$

$$\frac{\bar{X} - \mu}{s/\sqrt{n}} \stackrel{\sim}{\approx} N(0,1) \text{ if } n \text{ is large, but in small samples it isn't.}$$

William S. Gosset (Student), with degrees in both math and chem, obtained a post as a chemist with Arthur Guinness Son and Co in 1899. He invented the t-test to handle small samples for quality control in brewing.

3/3 He looked at

$$T = \frac{\bar{X}-\mu}{S/\sqrt{n}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{S/\sigma} = \frac{\overbrace{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}^{\sim N(0,1)}}{\sqrt{\underbrace{S^2/\sigma^2}_{\sim \chi^2}}}$$

The pdf of T is

$$f(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{\sqrt{(n-1)\pi}} \frac{1}{\left(\frac{t^2}{n-1}+1\right)^{n/2}}, \quad -\infty < t < \infty$$

↑ Gamma Functions

Testing for Normality

[Handout 8] 1. Check using Stem-and-Leaf Plot, Histogram, or Box Plot: Symmetric, unimodal, no outliers.

2. Look at Normal Probability Plot (also called Normal Scores Plot, Rankit Plot, and QQ Plot).

Assuming data comes from a $N(\mu, \sigma^2)$ look at ordered data.

Suppose data is $X_1, X_2, \ldots, X_n \Rightarrow$ ordered data: $\underset{\min}{X_{(1)}}, \underset{2^{nd}}{X_{(2)}}, \ldots, \underset{\max}{X_{(n)}}$.

Compare ordered values to what one would expect the ordered values to be if they truely came from a $N(\mu, \sigma^2)$.

→ Plot is a line through the origin with slope 1 ($y = x$).

Because we don't know μ, σ^2 (unknowns), we compare data with $N(0,1)$ values, giving a line with intercept μ and slope σ.

The close the points are to a line, the more likely the data have a normal distribution.

8.5 1/1

8.5 LARGE SAMPLE CI FOR POPULATION PROPORTION, p.

RECALL $X \sim Bin(n, p)$, X COUNTS THE NUMBER OF SUCCESSES OUT OF n INDEPENDENT TRIALS.

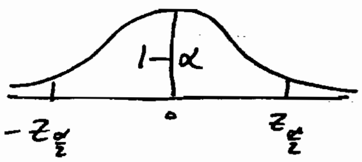
p = PROBABILITY OF SUCCESS.

THINK OF n AS THE SAMPLE SIZE FROM A LARGE POPULATION.

$$\left.\begin{array}{l} E(X) = np \\ Var(X) = np(1-p). \end{array}\right\} \quad Z = \frac{X - np}{\sqrt{np(1-p)}} \overset{a}{\sim} N(0,1), \text{ APPROXIMATELY.}$$

FACTOR OUT AN n, $Z = \dfrac{\frac{X}{n} - p}{\sqrt{\frac{p(1-p)}{n}}}$, LET $\hat{p} = \dfrac{X}{n}$ = SAMPLE PROPORTION.

THEN $P_r(-z_{\frac{\alpha}{2}} \le Z \le z_{\frac{\alpha}{2}}) = 1 - \alpha$



$$P_r\left(-z_{\frac{\alpha}{2}} \le \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \le z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$P_r\left(\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}} \le p \le \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}\right) = 1 - \alpha$$

DON'T KNOW POPULATION PROPORTION, p, SO ESTIMATE WITH $\hat{p}$.

$$P_r\left(\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right) = 1 - \alpha$$

A $100(1-\alpha)\%$ CI FOR p IS: $\hat{p} \pm z_{\frac{\alpha}{2}}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

PAR = p

EST = $\hat{p} = \dfrac{X}{n}$

SE(EST) = $\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

DIST OF $\dfrac{EST - PAR}{SE(EST)} \sim N(0,1)$

MARGIN-OF-ERROR

$ME = z_{\frac{\alpha}{2}}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$. FOR A GIVEN ME, CONFIDENCE LEVEL AND GUESSED/ESTIMATED PROPORTION, WHAT IS REQUIRED SAMPLE SIZE? USE CONSERVATIVE $\hat{p} = .5$

$n = \left(\dfrac{z_{\frac{\alpha}{2}}}{ME}\right)^2 \hat{p}(1-\hat{p})$ ↗ ROUNDED UP

ASSUMPTIONS: RANDOM SAMPLE OF n "INDIVIDUALS"

SAMPLE SIZE IS LARGE: $n\hat{p} > 5$ AND $n(1-\hat{p}) > 5$

\# OF SUCCESSES   # OF FAILURES

[HANDOUT 9]