

STAT 345 - Summer, 2006 - Exam 2

You may use a calculator and an 8½" by 11" sheet of paper.

Show all work to receive full credit.

1. A labeling machine damages the label on a jar of 505 Salsa with probability $p = 0.002$. Assume that jars are labeled independently of each other and let X denote the number of jars labeled until one is damaged.

- (a) What is the distribution of X ?

$$X \sim \text{geom}(0.002)$$

- (b) Find $P(X = 200)$.

$$f(x) = 0.998^{x-1}(0.002)$$

$$\Rightarrow P(X = 200) = 0.998^{199}(0.002) = 0.0013428$$

- (c) What is the expected number of jars labeled before one is damaged?

$$E(X) = \frac{1}{p} = \frac{1}{0.002} = 500 \text{ jars}$$

2. Let $X \sim \text{Binomial}(6, 0.75)$.

- (a) Find $E(X)$

$$E(X) = np = 6(0.75) = 4.5$$

- (b) Find $\text{Var}(X)$.

$$\text{Var}(X) = np(1-p) = 6(0.75)(0.25) = 1.125$$

- (c) Find $P(X = 2)$.

$$P(X = 2) = \binom{6}{2}(0.75)^2(0.25)^{6-2} = 0.03296$$

3. Let $X \sim \text{Bernoulli}(p)$; that is $P(X = 1) = p$ and $P(X = 0) = 1 - p$. From the definition of expected value, show that $E(X) = p$.
 $E(X) = 0(1 - p) + 1(p) = p$

4. A task force established by the EPA was scheduled to investigate 25 industrial firms to check for violations of pollution control regulations. However, budget cutbacks have drastically reduced the size of the task force and they will be able to investigate only four of the 25 firms (i.e. they had to sample 4, without replacement, from the 25). If it is known that six of the firms are actually operating in violation of regulations, find the probability that:

X = number of firms violating regulations in a sample of size $n = 4$ out of a population of size $N = 25$ where there are $K = 6$ firms known to be in violations $\Rightarrow X \sim \text{Hypergeometric}(25, 6, 4)$

- (a) None of the four sampled firms will be found in violation of regulations.

$$P(X = 0) = \frac{\binom{6}{0}\binom{19}{4}}{\binom{25}{4}} = 0.3064$$

- (b) All four firms investigated will be found in violation of regulations.

$$P(X = 4) = \frac{\binom{6}{4}\binom{19}{0}}{\binom{25}{4}} = 0.001186$$

- (c) At least one of the four firms will be operating in violation of pollution control regulations. (Set up but do not evaluate this probability.)

$$P(X \geq 1) = \sum_{x=1}^4 \frac{\binom{6}{x}\binom{19}{4-x}}{\binom{25}{4}}$$

5. Assume the number of earthquakes in the San Francisco Bay Area in a year is distributed Poisson with mean $\lambda = 12$.

(a) What is the probability that there are no earthquakes next year?

$$P(X = 0) = \frac{e^{-12}12^0}{0!} = 6.144 \times 10^{-6} = 0.000006144$$

(b) What is the probability that there are 2 or fewer earthquakes next year?

$$P(X \leq 2) = \frac{e^{-12}12^0}{0!} + \frac{e^{-12}12^1}{1!} + \frac{e^{-12}12^2}{2!} = 85e^{-12} = 0.0005222$$

6. The length of time for an individual to be served at a cafeteria is a random variable having an exponential distribution with a *mean* of 2 minutes.

(a) What is the probability that a person is served in less than 3 minutes?

$$P(X < 3) = \int_0^3 \frac{1}{2}e^{-\frac{x}{2}}dx = -e^{-\frac{x}{2}} \Big|_0^3 = 1 - e^{-\frac{3}{2}} = 0.7769$$

(b) What is the probability that a person is served in less than 3 minutes in *each* of the next 5 days assuming that days are independent?

Let $Y =$ the number of days a person is served in less than 3 minutes out of the next 5 days.

$$\text{Then } Y \sim \text{Bin}(5, 0.7769) \text{ and } P(Y = 5) = \binom{5}{5}0.7769^5(0.2231)^0 = 0.7769^5 = 0.28297$$

7. Suppose the continuous random variable X has the following probability density function:

$$f(x) = \begin{cases} kx^2 & -1 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

- (a) Find the value of k so that $f(x)$ is a pdf.

$$\int_{-1}^2 kx^2 dx = \frac{kx^3}{3} \Big|_{-1}^2 = \frac{k}{3}(8 - (-1)) = 3k = 1 \Rightarrow k = \frac{1}{3}$$

- (b) Find $P(X > 0.3)$.

$$P(X > 0.3) = \int_{0.3}^2 \frac{1}{3}x^2 dx = \frac{1}{9}x^3 \Big|_{0.3}^2 = \frac{8}{9} - \frac{0.3^3}{9} = 0.8859$$

- (c) Find $P(0.3 < X < 0.5)$.

$$P(0.3 < X < 0.5) = \int_{0.3}^{0.5} \frac{1}{3}x^2 dx = \frac{1}{9}x^3 \Big|_{0.3}^{0.5} = \frac{0.5^3}{9} - \frac{0.3^3}{9} = 0.0109$$

- (d) Find $E(e^X)$.

$$\frac{u}{x^2} \quad \frac{dv}{e^x}$$

$$\text{Using integration-by-parts: } \Rightarrow \begin{array}{ll} 2x & e^x \\ 2 & e^x \\ 0 & e^x \end{array}$$

$$\Rightarrow E(e^X) = \int_{-1}^2 e^x \frac{1}{3}x^2 dx = \left(\frac{1}{3}x^2 - \frac{2}{3}x + \frac{2}{3} \right) e^x \Big|_{-1}^2 = \frac{2}{3}e^2 - \frac{5}{3}e^{-1}$$

- (e) Find and sketch the CDF, $F(x)$.

$$\int_{-1}^x \frac{1}{3}t^2 dt = \frac{1}{9}t^3 \Big|_{-1}^x = \frac{1}{9}x^3 + \frac{1}{9}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{9}(x^3 + 1) & -1 \leq x \leq 2 \\ 1 & 2 < x \end{cases}$$

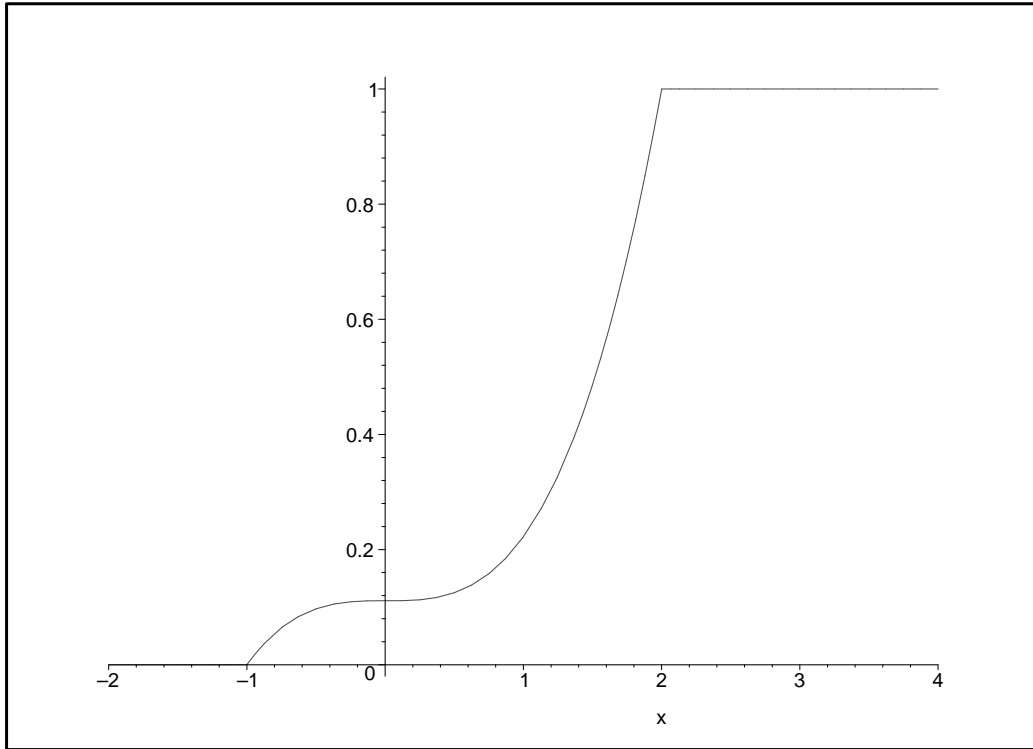


Figure 1: 7e. CDF

8. Suppose that we have two independent random variables, X_1 and X_2 . Further suppose that $E(X_1) = 1$, $Var(X_1) = 2$, $E(X_2) = 3$, and $Var(X_2) = 4$.

(a) Find the $E(2X_1 + 3X_2)$.
 $= 2(1) + 3(3) = 11$

(b) Find the $Var(X_1 + X_2)$.
 $= 2 + 4 = 6$

(c) Find the $Var\{2(X_1 - X_2)\}$.
 $= 4(2 + 4) = 24$

(d) Find the $E(X_1 - X_2)$.
 $= 1 - 3 = -2$

9. The loaves of rye-bread distributed to local stores by a certain bakery have an average length of 12 inches and a standard deviation of 0.5 inch. Assume that the lengths are normally distributed.

$$X \sim N(12, 0.5^2)$$

(a) What percentage of loaves are larger than 13 inches?

$$P(X > 13) = P\left(\frac{X - 12}{0.5} > \frac{13 - 12}{0.5}\right) = P(Z > 2) = 0.02275$$

(b) Suppose we need to order bags of the appropriate size for the loaves of bread. Find the length which we would expect 99 percent of the loaves to be under.

We want the value of x such that $P(X \leq x) = 0.99$
 $\Rightarrow P\left(\frac{X - 12}{0.5} \leq \frac{x - 12}{0.5}\right) = 0.99 \Leftrightarrow P(Z \leq z) = 0.99$ where
 $z = \frac{x - 12}{0.5}$

From the normal tables, $P(Z \leq 2.33) = 0.99 \Rightarrow 2.33 = \frac{x - 12}{0.5} \Rightarrow$
 $x = 13.165$ inches.