STAT 345 - Summer, 2005 - Practice Exam 2

BASED ON SECTIONS: 3.5 – 3.9, 4.1 – 4.7, 4.9, 5.1, 5.5, 5.7

1. (a) $X \sim Geom(0.98)$

   (b) $P(X = 1) = f(1) = (0.02)^{1-1}(0.98) = 0.98$

   (c) $P(X \leq 2) = f(1) + f(2) = 0.98 + (0.02)(0.98) = 0.9996$

   (d) $P(X > 2) = 1 - P(X \leq 2) = 1 - 0.9996 = 0.0004$

   (e) $E(X) = \frac{1}{0.98} = 1.02$ bottles

   (f) If $Y =$ the number filled until 2 are within tolerance then $Y \sim$ NegativeBin$(2, 0.98)$ and $E(Y) = \frac{2}{0.98} = 2.04$ bottles

2. (a) $X \sim Bin(6, 0.98)$

   (b) $E(X) = np = 6(0.98) = 5.88$ bottles

   (c) $P(X = 0) = \binom{6}{0} 0.98^0 (0.02)^6 = 6.4 \times 10^{-11}$

   (d) $P(X \leq 1) = P(X = 0) + P(X = 1) = 6.4 \times 10^{-11} + \binom{6}{1} 0.98 (0.02)^5 = 1.89 \times 10^{-8}$

3. (a) $X \sim$ Hypergeometric(20,5,10)

   (b) $E(X) = n\frac{K}{N} = 10\frac{5}{20} = 2.5$ white balls

   (c) $P(X = 0) = 0.0163$

   (d) $\{0, 1, 2, 3, 4, 5\}$

4. (a) $P(X \geq 3) = 1 - P(X < 3) = 1 - P(X \leq 2) = 1 - f(0) - f(1) - f(2) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!} - \frac{e^{-\lambda}\lambda^2}{2!} = 0.0803$ with $\lambda = 1$.

(b) Yes, as there is only an 8% chance of seeing three or more arrivals per minute.

5. $X \sim Bin(3, 0.5) \Rightarrow f(x) = \begin{pmatrix} 3 \\ x \end{pmatrix} 0.5^x (0.5)^{3-x}, x = 0, 1, 2, 3$
$\Rightarrow$

x	f(x)
0	0.125
1	0.375
2	0.375
3	0.125

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 0.125 & 0 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \\ 0.875 & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

For a graph, see class notes.

6. (a) $P(X > 3) = \int_3^\infty 2\mathrm{e}^{-2x} dx = \quad -\mathrm{e}^{-2x} \Big|_3^\infty = \mathrm{e}^{-6} = 0.0025$

(b) $P(3 < X < 4) = \int_3^4 2\mathrm{e}^{-2x} dx = \quad -\mathrm{e}^{-2x} \Big|_3^4 = -\mathrm{e}^{-8} + \mathrm{e}^{-6} = 0.0021$

(c) $P(X < x) = 1 - \mathrm{e}^{-2x} = 0.1$
$\Rightarrow 0.9 = \mathrm{e}^{-2x} \Rightarrow \ln(0.9) = -2x \Rightarrow x = -\frac{\ln(0.9)}{2} = 0.0527$

(d) See class notes.

(e) $F(x) = P(X \leq x) = \int_0^x 2\mathrm{e}^{-2t} dt = \quad -\mathrm{e}^{-2t} \Big|_0^x = 1 - \mathrm{e}^{-2x}$
$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 1 - \mathrm{e}^{-2x} & x \geq 0 \end{cases}$

7. (a) $f(x) = \frac{1}{2 - (-2)} = \frac{1}{4}, -2 \leq x \leq 2$

(b) $E(x) = \frac{a + b}{2} = \frac{-2 + 2}{2} = 0$
$Var(X) = \frac{(b - a)^2}{12} = \frac{[2 - (-2)]^2}{12} = \frac{16}{12} = \frac{4}{3}$

(c) $P(-x < X < x) = \frac{2x}{4} = \frac{x}{2} = 0.8 \Rightarrow x = 1.6$

(d) $E(e^x) = \int_{-2}^{2} \frac{e^x}{4} dx = \left. \frac{e^x}{4} \right|_{-2}^{2} = \frac{1}{4}(e^2 - e^{-2}) = 1.813$

(e) $E(X^2) = \int_{-2}^{2} \frac{x^2}{4} dx = \left. \frac{x^3}{3}\frac{1}{4} \right|_{-2}^{2} = \frac{8}{12} - \frac{-8}{12} = \frac{4}{3}$

Note that $Var(X) = E(X^2) - [E(X)]^2 = E(X^2)$ as $E(X) = 0 \Rightarrow E(X^2) = Var(X) = \frac{4}{3}$

8. (a) $E(X) = \int_{0}^{1} 3x^3 dx = \left. \frac{x^4}{4}\frac{1}{3} \right|_{0}^{1} = \frac{3}{4} = 0.75$

$E(X^2) = \int_{0}^{1} 3x^4 dx = \left. \frac{x^5}{5}\frac{1}{3} \right|_{0}^{1} = \frac{3}{5} = 0.6$

$\Rightarrow Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{5} - \left[\frac{3}{4}\right]^2 = \frac{3}{80} = 0.0375$

(b) $E(10X) = 10E(X) = 10(0.75) = 7.5$
$Var(10X) = 100Var(X) = 100(0.0375) = 3.75$

(c) $P(X > 0.5) = \int_{0.5}^{1} 3x^2 dx = \left. x^3 \right|_{0.5}^{1} = 1 - \frac{1}{8} = \frac{7}{8} = 0.875$

(d) $P(0.25 < X < 0.75) = \int_{0.25}^{0.75} 3x^2 dx = \left. x^3 \right|_{0.25}^{0.75} = (0.75)^3 - (0.25)^3 = 0.40625$

(e) $\int_{0}^{x} 3t^2 dt = \left. t^3 \right|_{0}^{x} = x^3$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

9. (a) $E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) = 2 + 5 + 10 = 17$

(b) $E(3X_1 + 2X_3) = 3E(X_1) + 2E(X_3) = 6 + 20 = 26$

(c) $Var(X_1+X_2+X_3) = Var(X_1)+Var(X_2)+Var(X_3) = 1+4+3 = 8$ as the RVs are independent.

(d) $Var(3X_1 + 2X_3) = 9Var(X_1) + 4Var(X_3) = 9 + 12 = 21$

(e) $Var(4(X_2 + X_3)) = 16(Var(X_2) + Var(X_3)) = (16)(7) = 112$

(f) $Var(2X_1 - 5X_3) = 4Var(X_1) + 25Var(X_3) = 79$

(g) $E(2X_1 - 5X_3) = 2E(X_1) - 5E(X_3) = 4 - 50 = -46$

10. $X \sim N(\mu, \sigma^2) = N(30, 4)$

    (a) $P(X > 31.7) = P\left(Z > \frac{31.7 - 30}{2}\right) = P(Z > 0.85) = 0.1977 \approx 20\%$

    (b) $P(29.3 < X < 33.5) = P\left(\frac{29.3 - 30}{2} < Z < \frac{33.5 - 30}{2}\right) = P(-0.35 < Z < 1.75) = 0.5968 \approx 60\%$

    (c) $P(X < 25.5) = P\left(Z < \frac{25.5 - 30}{2}\right) = P(Z < -2.25) = 0.0122 \approx 1.2\%$

    (d) Find x such that $P(X < x) = 0.99 \Leftrightarrow P\left(Z < \frac{x - 30}{2}\right) = 0.99 \Rightarrow \frac{x - 30}{2} = 2.326$ because the area to the left of $z = 2.326$ is 0.99 for $Z \sim N(0, 1) \Rightarrow x = 34.652$ cm.

11. (a) X = length of time in minutes until a randomly selected individual is served at a cafeteria

    $X \sim Exp(\lambda) \Rightarrow E(X) = \frac{1}{\lambda} = 4$ min $\Rightarrow \lambda = \frac{1}{4} = 0.25 \Rightarrow f(x) = 0.25\mathrm{e}^{-0.25x}, x \geq 0$

    P(served in less than 3 minutes) $= P(X < 3) = 1 - \mathrm{e}^{(-0.25)(3)} = 0.5276$

    (b) Let Y = the number of days out of the next 6 that a randomly selected person is served in less than 3 minutes. That is, Y counts the number of successes out of 6 where a success is being served in less than three minutes and the probability of a success is P(served in less than 3 minutes) = 0.5276 from part (a). $\Rightarrow Y \sim Bin(6, 0.5276) \Rightarrow$ the probability of being served in 4 of the next 6 days is $P(Y = 4) = \binom{6}{4} 0.5276^4 (1 - 0.5276)^2 = 0.2594$